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EE 364 - Fall 2026

Week 4

Supplemental Notes

Discrete random experiments

Defn: "Bernoulli trial" (a.k.a. coin flip)

binary outcome: H/T, T/F, 0/1, Success/Fail, ...

w/ $P[\text{success}] = p$.

Defn: A Bernoulli sequence $X_1, X_2, \ldots$ is a sequence of independent and identical Bernoulli trials

Defn: A sequence of outcomes $\{X_k\}$ is a random sample iff the outcomes are independent and identically distributed. (i.i.d.) **

*** random sample $\longleftrightarrow$ i.i.d.

Ex: 2 coin flips

$$P[H_1 \cap T_2] \overset{\text{indep.}}{=} P[H_1] \cdot P[T_2] \overset{\text{ident.}}{=} P[H] \cdot P[T] \quad (= p \cdot q)$$

Binomial probability "distribution"

$$p + \underbrace{(1-p)}_{q} = 1 \qquad 1 = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k \underbrace{(1-p)^{n-k}}_{q}$$

Idea: $\binom{n}{k} p^k (1-p)^{n-k}$

$p^k (1-p)^{n-k}$: prob of getting k "success" and n-k "fail" (ordered)

$\binom{n}{k}$: # different ways to arrange n "success" and n-k "fail" (n choose k)

$$P[\overbrace{\#\text{success}}^{X} = k] \text{ (in any order, in n trials)} = \binom{n}{k} p^k (1-p)^{n-k} \overset{\Delta}{=} b(n, k, p)$$

Binomial pdf (probability "mass" density function)

$$b(n,k,p) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{if} \quad k \le n \text{ and } p \in [0,1].$$

p = "success probability"
= $P[\text{Success on one Bernoulli Trial}]$.

$\therefore b(n,k,p) = P[k \text{ successes in } n\text{-trials}]$.

$$P[X \le k] = P[X=0] + P[X=1] + \cdots + P[X=k] = \sum_{j=0}^{k} b(n,k,p)$$

$$\therefore \sum_{k=0}^{n} b(n,k,p) = 1 \quad \text{by binomial theorem.}$$

since: $(p + (1-p))^n = 1^n = 1$.

Ex: $P[\underbrace{2}_{\text{means: exactly 2}} \text{ heads in 3 flips}] = P[X=2] = b(3,2,\underbrace{p}_{\text{Fair, 0.5}})$

$$= \binom{3}{2}(0.5)^2(0.5)^1 = \frac{3}{8}$$

$P[\text{at most 3 heads in } n \text{ flips}] = P[X \le 3]$

$$= \sum_{k=0}^{3} b(n,3,0.5)$$

$$= \sum_{k=0}^{3} \binom{n}{k}(0.5)^k(0.5)^{n-k}$$

$$= \frac{1}{2^n} \sum_{k=0}^{3} \binom{n}{k}$$

Ex: $P[3 \text{ heads in 5 flips}] = b(5,3,\tfrac{1}{2}) = \binom{5}{3}(\tfrac{1}{2})^3(\tfrac{1}{2})^2 = \boxed{5/16}$

$P[3 \text{ heads in 6 flips}] = b(6,3,\tfrac{1}{2}) = \binom{6}{3}(\tfrac{1}{2})^3(\tfrac{1}{2})^3 = \boxed{5/16}$

$P[2 \text{ heads in 5 flips}] = b(5,2,\tfrac{1}{2}) = \binom{5}{2}(\tfrac{1}{2})^2(\tfrac{1}{2})^3 = \boxed{5/16}$

Pay \$1 to play, \$3 win. Note: average win $\frac{15}{16} < 1$ "rational price"

$3 \cdot \frac{5}{16} + 0 \cdot \frac{11}{16}$ = expected return

Ex: Flip a coin n-times with $p = P(H)$. Suppose n is even. What is the probability of getting $\frac{n}{2}$ heads in n flips

Answer: $b(n, \frac{n}{2}, p) \approx \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} (4pq)^{n/2}$

$\therefore p = q = \frac{1}{2} \longrightarrow b(n,k,p) \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$

$\therefore$ Slow (square-root) decay.

computer: $P[X = n \text{ success in } 2n \text{ flips}] = \cdots = \binom{2n}{n} \frac{1}{2^{2n}}$

$\lim_{n \to \infty} \binom{2n}{n} \cdot \frac{1}{2^{2n}} = ? \qquad (= 0)$

so many ways of getting $\# \approx n$ but not n exactly. $\therefore P[X = n] \to 0$

Pf: $b(n,k,p) = \binom{n}{n/2} p^{n/2} q^{n - n/2}$ since n even

$= \frac{n!}{(n - n/2)!\,(n/2)!} (pq)^{n/2}$

$= \frac{n!}{((n/2)!)^2} (pq)^{n/2}$

$\approx \frac{\sqrt{2\pi n}\, \frac{n^n}{e^n}\, (p \cdot q)^{n/2}}{\left(\sqrt{2\pi \frac{n}{2}}\, \left(\frac{n}{2}\right)^{n/2} e^{-n/2}\right)^2}$ by Stirling's Approximation

$= \frac{\sqrt{2\pi n}\, \frac{n^n}{e^n}\, (pq)^{n/2}}{\pi \cdot n \cdot \frac{n^n}{2^n} e^{-n}}$

$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}\, 2^n\, (pq)^{n/2}$

$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}} (4 \cdot p \cdot q)$

$\therefore p = q = \frac{1}{2} \longrightarrow (4pq)^{n/2} = 1^{n/2} = 1$

$\therefore b(n,k,p) \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$

QED.

Ex: (computer)

$b(10,5,\frac{1}{2}) = 0.24609$ dbinom(5,10,0.5)

↑ faster decay if p≠q

Stirling: $b(10,5,\frac{1}{2}) \approx 0.25231$ sqrt(2/pi/10)

n	$b(n,\frac{n}{2},\frac{1}{2})$	Stirling's Approx.
100	0.03958	0.03978
1000	0.02522	0.02523
10,000	0.00798	0.00798
100,000	0.00252	0.00252

Compare to: (intuition) $\frac{\#\text{ success}}{\#\text{ total}} \longrightarrow p$

Multinomial probability

Trinomial : 3 outcomes "yes", "no", "abstain"

Put $Z = n - X - Y$ for n Bernoulli trials

with probabilities p_z, p_x, p_y : $p_x + p_y + p_z = 1$

$$\therefore 1 = 1^n = (p_x + p_y + p_z)^n = (\underbrace{p_x}_{a} + \underbrace{(p_y + p_z)}_{b})^n$$

B.T. $= \sum_{x=0}^{n} \binom{n}{x} p_x^x \underbrace{(p_y + p_z)^{n-x}}_{(a+b)^n}$

B.T. $= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \binom{n}{x}\binom{n-x}{y} p_x^x p_y^y p_z^{n-x-y}$

$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \frac{n!}{x!\cancel{(n-x)!}} \cdot \frac{\cancel{(n-x)!}}{y!(n-x-y)!} p_x^x p_y^y p_z^{n-x-y}$

$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \frac{n!}{x!\,y!\,(n-x-y)!} p_x^x p_y^y p_z^{\overbrace{n-x-y}^{z}}$

$= \sum_{\substack{x\geq 0, y\geq 0, z\geq 0 \\ x+y+z=n}} \frac{n!}{x!\,y!\,z!} p_x^x p_y^y p_z^z$

$\therefore$ trinomial pdf: $p(x,y) = P[X=x, Y=y]$

$$= \boxed{\frac{n!}{x!\,y!\,(n-x-y)!} p_x^x p_y^y p_z^{n-x-y}}$$

$\frac{n!}{x!\,y!\,(n-x-y)!} = \binom{n}{x,y,z}$ # ways to choose n w/ "x" Type 1, "y" Type 2 and "z" Type 3

Ex: How many ways to assign 20 students to groups A, B, C

w/ $k_A = k_B = 6$ and $k_C = 8$

$$\binom{20}{6,6,8} = \frac{20!}{6!\,6!\,8!} = 116,896,280$$

Note: $\underbrace{\binom{n}{k}}_{\text{binomial}} = \binom{n}{k,n-k}$

Ex: Sample w/ replacement

$p_x = 0.30$

$p_y = 0.15$ $\Big\}$ $p_z = 0.55$

① $P[$ 2 "X" and 3 "Y" if pick 10 $]$

$$\binom{10}{2,3,5}(0.30)^2(0.15)^3(0.55)^5$$

② $P[$ 2 "X" if pick 10 $]$ need "marginal" probability.

$p(x,y)$ describes "joint" probability of X and Y

$p(x)$ and $p(y)$ describe probability of X and Y seperately.

"marginal"

Marginal probability of trinomial

$$P(X=x) = \sum_{y=0}^{n-x} p(x,y) \overset{tri}{=} \sum_{y=0}^{n-x} \frac{n!}{x!\,y!\,(n-x-y)!} p_x^x p_y^y p_z^{n-x-y}$$

sample n, prob of x "X" regardless "Y" or "Z" comt

sum over all "y" w/ "x" fixed

$$= \frac{(n-x)!}{(n-x)!}\frac{n!}{x!} p_x^x \cdot \sum_{y=0}^{n-x} \frac{1}{y!\,(n-x-y)!} p_y^y p_z^{n-x-y}$$

$$= \frac{n!}{x!\,(n-x)!} p_x^x \sum_{y=0}^{n-x} \frac{(n-x)!}{y!\,(n-x-y)!} p_y^y p_z^{n-x-y}$$

$$= \binom{n}{x} p_x^x \left[\sum_{y=0}^{n-x} \binom{n-x}{y} p_y^y p_z^{(n-x)-y} \right]$$

B.T.

$$= \binom{n}{x} p_x^x (p_y + p_z)^{n-x} = \binom{n}{x} p_x^x (1-p_x)^{n-x}$$

($p_x + p_y + p_z = 1$)

$\therefore X \sim b(n, x, p_x)$

∴ marginal is binomial

Conditional probability of trinomial

$$P(X=x \mid Y=y) = p(x|y) = \frac{p(x,y)}{p(y)} = \frac{\frac{n!}{x!\,y!\,(n-x-y)!} p_x^x p_y^y p_z^{n-x-y}}{\frac{n!}{y!\,(n-y)!} p_y^y (1-p_y)^{n-x-y}}$$

Sample n, probability of x "X" given y "Y"

($\therefore n-y-x$ "Z")

$$= \frac{(n-y)!}{x!\,((n-y)-x)!} \cdot \frac{p_x^x \cdot p_z^{(n-y)-x}}{(1-p_y)^{n-y-x+x}} \quad (x+x=0)$$

$$= \binom{n-y}{x} \left(\frac{p_x}{1-p_y}\right)^x \left(1-\frac{p_x}{1-p_y}\right)^{(n-y)-x}$$

$$= b\left(n-y, x, \frac{p_x}{1-p_y}\right)$$

$\therefore$ conditional is binomial

Multinomial Theorem:

$$(p_1 + \cdots + p_k)^n = \sum_{l_1+\cdots+l_k=n} \frac{n!}{l_1!\,l_2!\cdots l_k!} p_1^{l_1} \cdots p_k^{l_k}$$

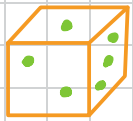
(partition)

if $l_1 + \cdots + l_k = n$

Compare: binomial : p_x, $p_y = 1-p_x$

trinomial : p_x, p_y, $p_z = 1-p_x-p_y$

Ex: - Roll a die ("d6") $\therefore l = 6$



- roll fair die 7 times, $\therefore n = 7$ $p_1 = \cdots = p_6 = \frac{1}{6}$

$$\therefore P(3\ 2\text{'s}, 2\ 4\text{'s}, 2\ 6\text{'s}) = \binom{7}{0,3,0,2,0,2} \left(\frac{1}{6}\right)^7$$

$$= \frac{7 \cdot 6 \cdot 5}{6^7} \approx 0.00078$$

- Deep neural classifier w/ K output SoftMax neurons (1-in-K coding)

$\therefore x \longrightarrow N \longrightarrow N(x)$, 1 roll of K-sided die ("categorical" multinomial)

$\therefore \ln p = \ln p_1^{x_1} \cdots p_K^{x_K} = \sum_{k=1}^{K} x_k \cdot \ln p_k$ — cross-entropy

$p = p_1^{x_1} \cdots p_K^{x_K}$; $p_1 \cdots p_K$ unknown ; $x_1 \cdots x_K$ data

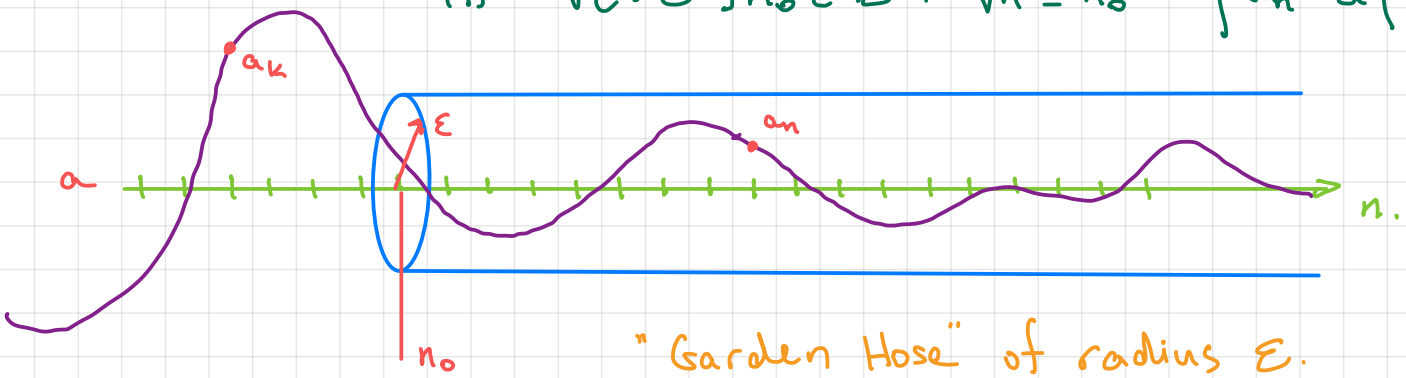
Q: find "best choice" of $p_1, \ldots, p_K$ given $x_1 \ldots x_K$ data/"observations"

Sequences and series

Defn: Limit of a sequence $a_1, a_2, a_3, \ldots$

$a_n \longrightarrow a$ iff $\lim_{n\to\infty} a_n = a$

iff $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 \quad |a_n - a| < \varepsilon$



"Garden Hose" of radius ε.

$\therefore$ smaller $\varepsilon > 0$ $\longrightarrow$ Takes longer to find n_0.

Partial Sum $\quad S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + \cdots + a_N$

Infinite Series $\quad$ "$\sum_{n=0}^{\infty} a_n = S$" converges

iff $\lim_{N\to\infty} S_N = S$ $\quad$ (S finite)

iff $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 : |S_N - S| < \varepsilon$.

(else "diverges")

Special case:

$\lim_{n\to\infty} a_n = \infty$ iff $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 \quad a_n > \varepsilon$.

- a type of non-convergence
- not the same in general as $\lim_{n\to\infty} a_n \neq a$.

Thm: (Triangle Inequality) $|a+b| \le |a| + |b|$

Prf: $|a+b|^2 = (a+b)^2 = a^2 + b^2 + 2ab$

$\le a^2 + b^2 + 2|a|\cdot|b|$ note: $|x| = \max(x, -x)$

$= |a|^2 + |b|^2 + 2|a|\cdot|b|$

$= (|a| + |b|)^2$

$\therefore |a+b| \le |a| + |b|$

Ex: **Thm:** $(a_n \to a) \wedge (b_n \to b) \implies (a_n + b_n \to a+b)$

Prf: Suppose $a_n \to a$ and $b_n \to b$

pick $\varepsilon > 0$

$\exists n_0 : \forall n \ge n_0 \quad |a_n - a| < \frac{\varepsilon}{2}$

$\exists m_0 : \forall n \ge m_0 \quad |b_n - b| < \frac{\varepsilon}{2}$

$\therefore \forall n \ge \max(n_0, m_0)$

$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)|$

$\overset{\text{T.I.}}{\le} |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

QED.

Thm: $|a-b| \ge ||a| - |b||$

Prf: $|a| - |b| = |(a-b) + b| - |b| \le |a-b| + |b| - |b| = |a-b|$

$\therefore |a| - |b| \le |a-b|$ similarly: $|b| - |a| \le |a-b|$

$\Downarrow$

$|a| - |b| \ge -|a-b|$

$\therefore \quad \underbrace{-|a-b|}_{-c} \le |a| - |b| \le \underbrace{|a-b|}_{c}$

$\therefore \boxed{||a| - |b|| \le |a-b|}$

QED.

Fact: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ where $e^x \triangleq \sum_{n=0}^{\infty}\frac{x^n}{n!}$

Fact: $\sum_{n=0}^{\infty} a_n = s \longrightarrow \lim_{n\to\infty} a_n = 0$

$(\not\Leftarrow)$

Test: p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$

- converges if $p > 1$.
- diverges if $p \le 1$

Ex: $\sum_{k=1}^{\infty}\frac{1}{n}$ diverges since $\frac{1}{n} = \frac{1}{n^1}$, $\therefore p = 1$

$\sum_{k=1}^{\infty}\frac{1}{n^2}$

$\sum_{k=1}^{\infty}\frac{1}{n^{1.0001}}$

$\Big\}$ converge

$n = 2 > 1$

$n = 1.0001 > 1$

Test: (Alternating Series test)

$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if

- $a_k \ge a_{k+1} > 0 \quad \forall k \in \mathbb{Z}^+$.
- $\lim_{n\to\infty} a_n = 0$

Ex: $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}$ converges since: 1) $\frac{1}{n} \ge \frac{1}{n+1}$ $\forall n$

2) $\lim_{n\to\infty}\frac{1}{n} = 0$.

Defn: $\sum_{n=0}^{\infty} a_n$ <u>converges absolutely</u> iff $\sum_{n=0}^{\infty}|a_n|$ converges

↑ ($\therefore$ treat infinite sums like finite sums)

Facts: (1) Absolute convergence $\longrightarrow$ convergence ($\not\Longleftarrow$)

(2) If
- $\sum_{n=1}^{\infty} a_n$ converges absolutely
- $\sum_{n=1}^{\infty} b_n$ is any rearrangement of $\sum_{n=1}^{\infty} a_n$

Then:
- $\sum_{n=1}^{\infty} b_n$ converges
- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Defn: $\sum_{n=0}^{\infty} a_n$ converges conditionally iff
- $\sum_{n=0}^{\infty} a_n$ converges
- $\sum_{n=0}^{\infty} |a_n|$ diverges

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges conditionally since $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges but $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(extra, not tested)

Thm: (Riemann's Rearrangement Theorem) Pick ANY $x \in \mathbb{R}$. Suppose $\sum_{n=0}^{\infty} a_n$ converges conditionally. Then $\exists$ a re-arrangement $\sum_{n=0}^{\infty} b_n$ of $\sum_{n=0}^{\infty} a_n$ where $\sum_{n=0}^{\infty} b_n = x$.

(example: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

Ratio test for convergence of $\sum_{n=0}^{\infty} a_n$ (series)

Put $L \overset{\Delta}{=} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ Then

(1) $L < 1 \Longrightarrow$ series converges absolutely.

(2) $L > 1 \Longrightarrow$ series diverges

(3) $L = 1 \Longrightarrow$ test fails

Ex: Geometric Series: $a_n = a^n$. $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a^n$ $(a \neq 0)$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{a^{n+1}}{a^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{a^n \cdot a}{a^n}\right|$$

$$= \lim_{n\to\infty} |a|.$$

$$= |a|.$$

$\therefore \sum_{n=0}^{\infty} a^n$ converges absolutely if $|a| < 1$.

Ex: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{1}{n+1}\right|$$

$$= |x| \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0 \qquad \therefore e^x \text{ converges } \forall x.$$

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/n+1}{1/n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1.$

$\therefore$ test fails.

Fact: $|x| \leq c$ iff $-c \leq x \leq c$ $(c > 0)$



Note: Taylor's series

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(0)}{n!}}_{a_n} \cdot x^n$$

i.e. power series "around 0"

Power Series for variable x

$$\sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Ex: Find all x so that $\sum_{n=0}^{\infty} \frac{n}{5^n} x^n$ converges absolutely.

By the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{n \cdot x^n} \cdot \frac{5^n}{5^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)x}{5n} \right|$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)}_{=1} \cdot \frac{1}{5} \cdot |x|.$$

$$= \frac{1}{5}|x|$$

$$< 1$$

iff $|x| < 5$. iff $-5 < x < 5$

1. $x = +5$ $\quad \sum_{n=0}^{\infty} \frac{n}{\cancel{5^n}} \cdot \cancel{5^n} = \sum_{n=0}^{\infty} n$ diverges

2. $x = -5$ $\quad \sum_{n=0}^{\infty} \frac{n}{5^n} \cdot (-5)^n = \sum_{n=0}^{\infty} (-1)^n \cdot n$ diverges

$\therefore$ $\sum_{n=0}^{\infty} \frac{n}{5^n} x^n$ converges iff $-5 < x < 5$ (else diverges)

Thm: (1) If $\sum_{n=0}^{\infty} a_n x^n$ converges for $c \neq 0$ then it converges absolutely $|x| < |c|$.

(2) If $\sum_{n=0}^{\infty}$ diverges for $d \neq 0$ then it diverges if $|x| > |d|$.

Thm: Exactly one holds for $\sum_{n=0}^{\infty} a_n x^n$

(1) it converges only if $x = 0$

(2) it converges absolutely for all x (e.g. e^x)

(3) $\exists r > 0$: { It converges absolutely if $|x| < r$. and It diverges if $|x| > r$.

$\therefore$ Case 3: "radius of convergence" $r > 0$

$(-r, r)$

$X \sim NB$: Negative Binomial

$P(X = n \text{ trials until } k \text{ successes})$

$= P(\text{head on } n^{\text{th}} \text{ flip}) \cdot P(k-1 \text{ heads in first } n-1 \text{ flips})$

(n^{th} trial; first $n-1$ trial; independent)

$= p \cdot \binom{n-1}{k-1} p^{k-1} \cdot (1-p)^{(n-1)-(k-1)}$

$= \boxed{\binom{n-1}{k-1} p^k (1-p)^{n-k}}$

$X \sim G(p)$: Geometric

$P(X = n \text{ trials until } 1^{\text{st}} \text{ success})$

$\therefore = NB(n, k=1)$

$= \binom{n-1}{0} p^1 (1-p)^{n-1}$

$= \boxed{p \cdot (1-p)^{n-1}}$

Thm: $\sum_{k=0}^{n} a^k = \begin{cases} n+1 & \text{if } a=1 \\ \frac{1-a^{n+1}}{1-a} & \text{if } a \neq 1. \end{cases}$

Prf: $S \triangleq a^0 + a^1 + \cdots + a^n = \sum_{k=0}^{n} a^k$

$\therefore aS = a + a^2 + \cdots + a^{n+1}$

$\therefore S(1-a) = S - a \cdot S$

$= (1 + a + \cdots + a^n) - (a + a^2 + \cdots + a^{n+1})$

$= 1 - a^{n+1}$

$\therefore S = \frac{1-a^{n+1}}{1-a}$ since $a \neq 1$

QED

Corr 1: $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ if $|a| < 1$

Corr 2: $\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$ if $|a| < 1$

Corr 3: $\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a}$ if $|a| < 1$

Q: verify geometric is a "valid" probability

$$\sum_{\Omega} p(x) \overset{G(p)}{=} \sum_{n=1}^{\infty} p \cdot (1-p)^{n-1} = p \cdot \sum_{n=1}^{\infty} (1-p)^{n-1}$$

Let $k = n-1$, $\therefore n = k+1$

$$= p \cdot \sum_{k+1=1}^{k+1=\infty} (1-p)^k$$

$$= p \sum_{k=0}^{\infty} (1-p)^k \qquad 0 \leq 1-p < 1$$

$$= p \cdot \frac{1}{1-(1-p)}$$

$$= p \cdot \frac{1}{p} = 1$$

$\therefore$ "valid" probability

Thm: $P(X > k) = q^k$ if $X \sim G(p)$

$\therefore P(X \leq k) = 1 - q^k$

Prf: $P(X > k) = P(\{X = k+1\} \cup \{X = k+2\} \cup \cdots)$

("OR" between the events)

$\overset{\text{CAT}}{=} \sum_{x=k+1}^{\infty} p(x)$

$\overset{X \sim G(p)}{=} \sum_{x=k+1}^{\infty} p \cdot q^{x-1} \overset{\substack{y = x-1 \\ \therefore x = y+1}}{=} p \cdot \sum_{y=k}^{\infty} q^y$

$\overset{\text{Corr 3}}{=} p \cdot \frac{q^k}{1-q} = \frac{p}{p} \cdot q^k = q^k$ QED.

Fact: $\binom{N}{n} = \sum_{k=0}^{n} \binom{N_1}{k}\binom{N_2}{n-k}$ (sum to N; sum to n) if $N = N_1 + N_2$ and $\binom{k}{j} = 0$ for $j > k$.

(N: total pop.; N_1: # type 1; N_2: # type 2)

Hypergeometric probability

Suppose batch of N items. Sample n w/o replacement.

(N_1 type 1, N_2 type 2)

$$\binom{N}{n} : \binom{N_1 + N_2}{n} = \sum_{k=0}^{n} \binom{N_1}{k}\binom{N_2}{n-k}$$

$\binom{N_1}{k}\binom{N_2}{n-k}$: # different ways to draw k "good" (Type 1) and $n-k$ "bad" (Type 2)

$\sum_{k=0}^{n}$: add over all ways for 0 "good", 1 "good", ... n "good"

$$\frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}} \geq 0 \quad \text{and} \quad \sum_{k=0}^{n} \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}} = 1.$$

$\therefore \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}$ obeys all axioms of probability.

w/ $P[k$ "good" from draw of $n]$

hypergeometric p.m.f.

Ex: small lake, $N = 50$ fish → $N_1 = 10$ trout

→ $N_2 = 40$ large mouth bass

Bob catches: 7 fish w/ replacement ("catch and release")

$$P[\text{2 trout and 5 bass}] = b\left(7, 2, \tfrac{1}{5}\right)$$
$$= \binom{7}{2}\left(\tfrac{1}{5}\right)^2\left(\tfrac{4}{5}\right)^5$$
$$= 0.275$$

7 fish w/o replacement ("catch and eat")

$$P[\text{2 trout and 5 bass}] = \text{Hypergeo}\left(\overset{n}{7}, \overset{N_1}{10}, \overset{k}{2}, \overset{N}{50}\right)$$
$$= \frac{\binom{10}{2}\binom{40}{5}}{\binom{50}{7}}$$
$$= 0.2969$$

Q: probability w/ replacement and w/o replacement are similar. Under what conditions is this a "good" approximation?

(i.e. when can you act as if "with replacement")

A: if the pond is an ocean. → N big, n small

(fixed ratio $\frac{N_1}{N}$)

Binomial approximation to the hypergeometric

Thm Hypergeometric $\xrightarrow{d}$ Binomial if $N \to \infty$ and $p = \frac{n_1}{N}$

Prf: Say $X \sim Hyp(n, m, k, N)$. Fixed k & n

$$\therefore \lim_{N\to\infty} P(X=k) = \lim_{N\to\infty} \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

$$= \lim_{N\to\infty} \frac{m!}{k!(m-k)!} \cdot \frac{(N-n)!\,n!}{N!} \cdot \frac{(N-m)!}{(n-k)!(N-n-m+k)!}$$

$$= \lim_{N\to\infty} \frac{\frac{n!}{(n-k)!\,k!} \cdot \frac{m!(N-m)!}{(N-m-n+k)!(m-k)!}}{\frac{N!}{(N-n)!}}$$

$$= \lim_{N\to\infty} \binom{n}{k} \frac{\frac{m!}{(m-k)!} \cdot \frac{(N-m)!}{(N-m-n+k)!}}{\frac{N!}{(N-n)!}}$$

$$= \lim_{N\to\infty} \frac{\prod_{j=1}^{k}(m-k+j)\prod_{j=1}^{N-k}(N-m-n+k+j)}{\prod_{j=1}^{n}(N-n+j)} \cdot \frac{N^n}{N^n}$$

$\binom{n}{k}$ fixed

$$= \binom{n}{k} \cdot \lim_{N\to\infty} \frac{\prod_{j=1}^{k}\left(\underbrace{\frac{m}{N}}_{p} - \frac{k}{N} + \frac{j}{N}\right)\prod_{j=1}^{n-k}\left(\underbrace{1-\frac{m}{N}}_{1-p} - \frac{n}{N} + \frac{k}{N} + \frac{j}{N}\right)}{\prod_{j=1}^{n}\left(1 - \frac{n}{N} + \frac{j}{N}\right)}$$

$$= \binom{n}{k} \cdot \left(\frac{m}{N}\right)^k \left(1-\frac{m}{N}\right)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= b(n, k, p)$$

QED

BEG CUP

Sampling

Outcomes	w/ replacement	w/o replacement
2	Binomial; negative binomial / geometric ("until")	Hypergeometric
≥ 2	Multinomial	Multivariate Hypergeometric

Hypergeo → Binomial

Multi Hypergeo → Multinomial

$$\frac{\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_K}{n_K}}{\binom{N}{n}} \qquad n = n_1 + \cdots + n_K \quad N = N_1 + \cdots + N_K$$

Issue-Spotting Sequence

Q1: Outcomes? (2 or more?)

Q2: Sampling? (w/ or w/o replacement)

Q3: Until structure?

"UNTIL" ⟹ negative binomial

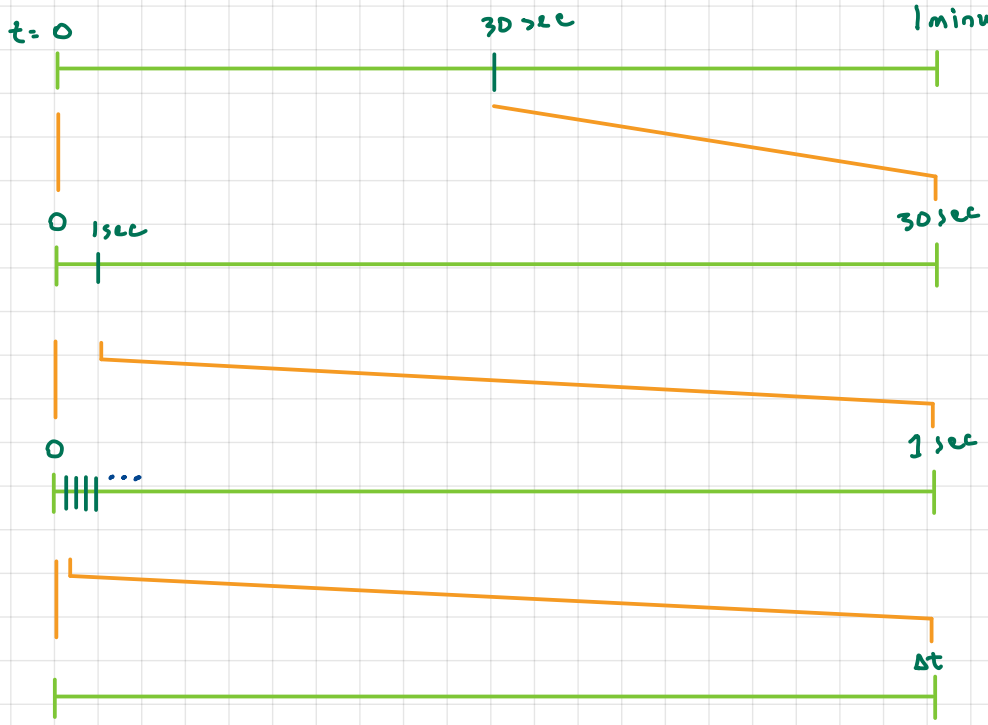
(k=1 outcomes, geometric)

UNTIL

- Discrete → Negative Binomial (k=1: Geometric)
- Continuous → γ (gamma) (k=1: Exponential; k>1: Erlang ($k \in \mathbb{Z}^+$))

Poisson probability — counting independent arrivals.

Suppose on average 120 packets / minute



avg: 120 packets/min

avg: 60 packets/30 sec

avg: 2 packets/sec.

avg: <<< 1 packet / Δt.

"rate" remains constant

$\lambda = 120 \frac{1}{\text{min}} \cdot \left(\frac{1}{\Delta t}\right) \cdot \Delta t$

n intervals

$\therefore$ very small $\Delta t \longrightarrow P[\text{1 arrival in interval}] = p. <<< 1 \quad (\approx 0)$

$\therefore P[\text{2 arrivals}] = p^2, \; P[\text{3 arrivals}] = p^3, \ldots$

independent arrivals

$\therefore \Delta t \longrightarrow 0, \; P[\text{1 arrival}]$ small, $P[\text{more than 1 arrival}] \approx 0.$

$\therefore$ binomial structure: n flips, $p[\text{1 arrival}] = p$, with $n \cdot p = \lambda$.

large — small — constant rate

Thm (Poisson Law) $b \xrightarrow{d} p$ if $n \gg 1$ and $p \ll 1$ and $\lambda = np$.

Pf: $\lim_{n\to\infty} b(n,k,p) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$.

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n-k} \quad \text{if } \lambda = np.$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\, n^k} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n\cdot(n-1)\cdots(n-k+1)}{n \;\; n \;\; \cdots \;\; n} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} 1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k+1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \underbrace{\lim_{n\to\infty}\left(1-\frac{1}{n}\right)}_{=1} \cdots \underbrace{\lim_{n\to\infty}\left(1-\frac{k+1}{n}\right)}_{=1} \cdot \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \cdot \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k}}_{=1}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= P(\lambda)$$

QED.

Poisson pdf

BEG CUP

issue spot: counting structure

$X \sim P(\lambda)$ for parameter ("mean") $\lambda > 0$

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

Ex: Alpha-decay of Radium to Radon

$$^{226}_{88}Ra \longrightarrow {}^{222}_{86}Rn + {}^{4}_{2}He^{2+}$$

($^{4}_{2}He^{2+}$: α-particle)

half-life of ^{226}Ra isotope: 1600 years

Radium: 226 grams / mole

$\therefore$ 1 gram $\approx 10^{22}$ atoms $(= n)$

Binary event: atom "decays" spontaneously or not.

"Vast inter-atomic distance" $\longrightarrow$ independence

$\lambda =$ α-emissions per-gram per-second

$\approx 10^{10}$ decays

$T =$ 1 second

$$\therefore P(1\ \alpha\text{-decay}) = b(10^{25}, 1) \approx \frac{10^{10}}{10^{22}} = 10^{-12}$$

$$= \frac{1}{\text{trillion}}$$

$$\therefore P(X(t) = k) \approx \frac{e^{-\lambda}\lambda^{k}}{k!} \quad \text{for } k = 0, 1, 2, \ldots$$

Ex: Basketball long shots $\quad X \sim b(70, 0.02)$

assume independent shots

$\therefore$ $n = 70 \gg 1$, $p = 0.02 \ll 1$

$$\therefore \lambda = n \cdot p = 70 \cdot (0.02) = 1.4$$

$\therefore$ Exact binomial probability

$$P(3 \text{ baskets in } 70 \text{ trials}) = \binom{70}{3}(0.02)^{3}(0.98)^{67}$$

$$= 0.1131$$

∴ Poisson approximation with $\lambda = np = 1.4$

$$P(X=3) = \frac{(1.4)^3 e^{-1.4}}{3!} = 0.1128$$

Later $b(n,k,p) \longleftrightarrow M_B(s) = (1-p+pe^s)^n$ ("moment generating function")

$$\left(= E[e^{sX}]\right)$$

$$P(\lambda) \longleftrightarrow M_P(s) = e^{\lambda(e^s-1)}$$

∴ For $n \cdot p = \lambda$ constant.

$$\lim_{n\to\infty} M_B(s) = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^s\right)^n \qquad \text{since } p = \frac{\lambda}{n}$$

$$= \lim_{n\to\infty} \left(1 - \frac{\lambda(e^s-1)}{n}\right)^n$$

$$= e^{\lambda(e^s-1)}$$

$$= M_P(s)$$

∴ Again: $b(n,k,p) \xrightarrow{d} P(\lambda)$ if $\lambda = n \cdot p$ (and Levy's Theorem, week 11)

Accuracy of Poisson Approximation

Okay:	$n \geq 20$	and	$p \leq 0.05$
Excellent:	$n \geq 100$	and	$p \leq 0.1$